FREE FERMIONS REVISITED

ANTUN MILAS, MICHAEL PENN

Abstract. Free fermion vertex superalgebras are discussed from the point of view of Urod vertex algebras [3, 4]. We present all finite decompositions of the 3-fermion vertex algebra via Virasoro and $N = 1$ superconformal vertex algebras. We also present decompositions of higher rank fermion algebras using affine W-algebras, and a conjecture on the existence of the "square root" of the $(2n + 1)$ fermion algebra.

1. Introduction

The free fermion vertex algebra is, in a suitable sense, the smallest rational vertex (super)algebra, holding a pivotal role within the theory of vertex algebras. For some basic facts about free fermions, charged free fermions, and the celebrated Boson-Fermion correspondence see [7, 10]. To introduce this algebra, we first define the fermionic superalgebra with odd generators φ_n, $n \in \mathbb{Z} + \frac{1}{2}$, obeying anti-commutation relations:

$$[\varphi_m, \varphi_n]_+ = \delta_{m+n,0}.$$

Then the free fermion vertex algebra (also known as 'fermionic Fock space') is given by $\mathcal{F} = \Lambda^*(V)$ where $\Lambda^*(V)$ denotes the exterior algebra on $V = \text{Span}\{\varphi_{-1/2}, \varphi_{-3/2}, \ldots\}$. It is well known that $\mathcal{F}$ can be equipped with a unique conformal vertex algebra structure of central charge $\frac{1}{2}$ with conformal vector $\omega_{\mathcal{F}} = \frac{1}{2}\varphi_{-3/2}\varphi_{-1/2}\mathbf{1}$ (throughout, $\mathbf{1}$ denotes the vacuum vector). Under this conformal structure, the fermionic generator $\varphi := \varphi_{-1/2}\mathbf{1}$ has degree $\frac{1}{2}$. The space $\mathcal{F}$ further decomposes as a Virasoro algebra module (notation defined below) [10]:

$$\mathcal{F} = L\left(\frac{1}{2}, 0\right) \oplus L\left(\frac{1}{2}, \frac{1}{2}\right).$$

Clearly, the character [1] of $\mathcal{F}$ with respect to $L(0) := (\omega_{\mathcal{F}})_{(1)}$ is given by

$$\text{ch}[\mathcal{F}](q) = (-q^{\frac{1}{2}}; q)_\infty,$$

where we use q-Pochammer symbol notation $(a; q)_n := \prod_{i=1}^n (1 - aq^{i-1})$.

More generally, we consider the tensor product of n copies of $\mathcal{F}$, $\mathcal{F}^{\otimes^n} := \mathcal{F} \otimes \cdots \otimes \mathcal{F}$, of central charge $c = \frac{n}{2}$ with the character $\text{ch}[\mathcal{F}^{\otimes^n}](q) = (-q; q^{1/2})^n_\infty$. These vertex superalgebras are important as they carry actions of the level one orthogonal affine Lie algebras [7], (see also [14]). Moreover, modules of $L(\frac{1}{2}, 0)^{\otimes^n}$ are building blocks of *holomorphic* vertex algebras, including the Moonshine Module. Observe that with $c = \frac{n}{2}$, the character $q^{-c/24}\text{ch}[\mathcal{F}^{\otimes^n}]$ is a modular form of weight zero (notice the n-th power of Weber's modular functions $\mathfrak{f}(q) = q^{-1/48}(-q^{1/2}; q)_\infty)$).

[1] Here the character is defined as Tr $q^{L(0)}$, where $L(0)$ is the degree operator.

In this paper we are interested in a new grading on $\mathcal{F}^{\otimes^n}$, specifically pertaining to integral shifts of fermionic modes, such that for $n \geq 1$:

$$\mathrm{ch}_{new}[\mathcal{F}^{\otimes^{2n+1}}](q) = \prod_{i=-n}^{n} \prod_{m\geq 1}^{\infty} (1+q^{m-i-1/2}) = q^{-\sum_{i=1}^{n} \frac{i^2}{2}} \prod_{m\geq 1}^{\infty} (1+q^{m-1/2})^{2n+1},$$

and for $n \geq 2$,

$$\mathrm{ch}_{new}[\mathcal{F}^{\otimes^{2n}}](q) = (-q^{1/2};q)_\infty \prod_{i=-(n-1)}^{n-1} \prod_{m\geq 1}^{\infty} (1+q^{m-i-1/2}) = q^{-\sum_{i=1}^{n-1} \frac{i^2}{2}} \prod_{m\geq 1}^{\infty} (1+q^{m-1/2})^{2n}.$$

Observe that the conformal weights (or degrees) of generating fields receive both positive and negative integral shifts. If we try to turn these modified characters into modular functions using the standard procedure, $q^{-c/24}\mathrm{ch}[\cdot]$, a new central charge arises and we require

$$c_{\mathcal{F}^{\otimes 2n+1}} = -24\left(\sum_{i=1}^{n} \frac{i^2}{2} - \frac{2n+1}{48}\right) = -\frac{1}{2}(2n+1)(4n^2+4n-1),$$
$$c_{\mathcal{F}^{\otimes 2n}} = -24\left(\sum_{i=1}^{n-1} \frac{i^2}{2} - \frac{2n}{48}\right) = -n(1-6n+4n^2).$$

As we shall see in Section 8, these are precisely *Urod* central charges of the $\ell = 1$ level simple affine vertex algebra of $\mathfrak{so}(2n+1)$ [2] and $\mathfrak{so}(2n)$, respectively. The Urod vertex algebra of $\mathfrak{sl}_2$ with $\ell = 1$ was introduced by Bershtein, Feigin, and Litvinov in [4] and studied in full generality for all simple Lie algebras in [3]; see also [6] for their relevance to gluing operations on 4-manifolds and CFT.

The objective of this paper is to examine various decompositions of $\mathcal{F}^{\otimes^n}$ featuring a distinctive grading as above, in terms of affine W-algebras with a specific focus on the $n = 3$ case.

Our paper is structured as follows. In Section 2, we compile known facts concerning Lie algebras, vertex algebras, and their representations pertinent to our study. In Section 3, we study a family of (super)conformal structures on the 3-fermion vertex algebra $\mathcal{F}^{\otimes^3}$, denoted by $\mathcal{U}_{N=1}$; see formulas (5),(6). Then we obtain all finite decompositions of $\mathcal{U}_{N=1}$ in terms of $(S)Vir \times (S)Vir$-modules (see Proposition 3.2, Theorem 3.3, etc.). Additionally, we discuss the 2-fermion model; see Proposition 3.6. In Section 4, which is mostly a consequence of results from [3] and [12], we give a decomposition of $L(c_{p,p'}, 0) \otimes \mathcal{U}_{N=1}$ vertex algebra in Theorem 4.4. Furthermore, in Section 4.5 we furnish several decompositions for $L_{ns}(c_{p,p'}, 0) \otimes \mathcal{U}_{N=1}$ for special central charges. Section 5 is devoted to proving that the family of Urod conformal structure introduced in Section 3 also includes the special one introduced in [3]. In Section 6, we present a decomposition formula for the 4-fermion algebra, and in Section 7, we propose conjectures concerning the rank five fermionic algebra. In Section 8, we consider the general case $\mathcal{F}^{\otimes^n}$ and related decompositions including Conjecture 8.3 pertains to a "square root" vertex subalgebra of $\mathcal{F}^{\otimes^{2n+1}}$. The concluding part includes explicit calculations of conformal vectors needed for Theorem 4.2. We finish with brief remarks for future work.

2. Preliminary results

2.1. **Affine vertex algebra of $\mathfrak{sl}(2)$.** Let e, f, h denote the standard generators of $\mathfrak{sl}_2$ with bracket relations $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. We choose the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$ and equip

[2] For $n = 1$, $\mathfrak{so}(3) \cong \mathfrak{sl}_2$ so the affine Lie algebra level for $\mathcal{F}^{\otimes^3}$ is 2.

$\mathfrak{sl}_2$ with an invariant, non-degenerated bilinear form such that $(h,h)=2$. We denote the positive simple root by α and the fundamental weight by $\omega_1=\frac{\alpha}{2}$.

Denote by $\widehat{\mathfrak{sl}}_2$ the affine Lie algebra spanned by $x_n = x\otimes t^n$, $n\in\mathbb{Z}$, and central element C, with bracket relations (here $x,y\in\mathfrak{sl}_2$)

$$[x_n,y_m]=[x,y]_{n+m}+n\delta_{m+n,0}(x,y)C.$$

Then the induced module $V_{\mathfrak{sl}_2}(k,0)\cong U(t^{-1}\mathfrak{sl}_2[t^{-1}])$ with level $k\neq -2$, has a vertex operator algebra structure called the vacuum module. The Sugawara conformal vector (for $k\neq -2$) is given by

$$\omega_{Sug}=\frac{1}{2(k+2)}\left(e_{-1}f_{-1}\mathbf{1}+f_{-1}e_{-1}\mathbf{1}+\frac{1}{2}h_{-1}^2\mathbf{1}\right),$$

with central charge $c=\frac{3k}{k+2}$. Denote the simple quotient of $V_{\mathfrak{sl}_2}(k,0)$ by $L_{\mathfrak{sl}_2}(k,0)$. For $k\in\mathbb{N}$, this vertex algebra is rational and a complete set of irreducible $L_{\mathfrak{sl}_2}(k,0)$-modules is given by $L_{\mathfrak{sl}_2}(k,n\omega_1)$, with $0\leq n\leq k$. If $k=-2+\frac{p}{p'}$ where p,p' are coprime and $p\geq 2$, the level is called admissible and all irreducible admissible modules in category $\mathcal{O}$ are given by: $L_{\mathfrak{sl}_2}(k,((m-1)-(k+2)(m'-1))\omega_1)$, where $0<m<p$ and $0<m'<p'$.

More generally, $V_{\mathfrak{g}}(k,0)$, with level $k\neq -h^\vee$, will denote the vacuum affine vertex algebra associated to a simple Lie algebra $\mathfrak{g}$ and $V_{\mathfrak{g}}(k,\lambda)$ its module with highest weight $\lambda\in\mathfrak{h}^*$. The corresponding simple vertex algebra and modules will be denoted by $L_{\mathfrak{g}}(k,0)$ and $L_{\mathfrak{g}}(k,\lambda)$, respectively.

2.2. **Virasoro algebra.** Let Vir denotes the Virasoro Lie algebra, spanned by L_n, $n\in\mathbb{Z}$, and C obeying the usual bracket relation. Irreducible lowest weight module of lowest weight h and central charge c will be denoted by $L_{Vir}(c,h)$, or simply $L(c,h)$ when the context implies the Lie algebra is Vir. The Verma module will be denoted by $M(c,h)$ and we also let $V(c,0)=M(c,0)/(L(-1)v_{c,0})$, the vacuum vertex algebra. It is convenient to use the following parametrization for the central charge ($t\neq 0$):

$$c_t=1-6(t+t^{-1}-2).$$

We let $h_{\alpha,\beta}=\frac{t}{4}(\alpha^2-1)-\frac{1}{2}(\alpha\beta-1)+\frac{1}{4t}(\beta^2-1)$, where $\alpha,\beta\in\mathbb{Z}$. If $\alpha,\beta\in\mathbb{Z}_{>0}$ then $M(c_t,h_{\alpha,\beta})$ is reducible as shown by Feigin and Fuchs (cf. [8] for more detailed account). Moreover, $V(c_t,0)$ is simple if and only if $t\neq\frac{p}{p'}$ where p and p' are integers ≥ 2 and $\gcd(p,p')=1$. The case $t\neq\frac{p}{p'}$ is what we will refer to as *generic* central charge. For generic central charges we shall need the following result:

Proposition 2.1. *Let $c=c_t$ be generic. Then $M(c_t,h)$ is reducible for $h=h_{1,n}=\frac{(n-1)(n+1)}{4t}-\frac{n-1}{2}$. For this value of h, there is a singular vector of weight n in $M(c,h_n)$ that generates the maximal submodule. Consequently,*

$$\mathrm{ch}[L(c_t,h_{1,n})](q)=q^{h_{1,n}}\frac{(1-q^n)}{(q;q)_\infty}.$$

On the opposite spectrum, for $t=\frac{p}{p'}$, the vertex algebra $L(c_{p/p'},0)$ is rational. Moreover, Virasoro modules $L(c_{p/p'},h_{r,s})$, also denoted by $L(c_{p,p'},h_{r,s})$, where $1\leq r\leq p-1, 1\leq s\leq p'-1$ and $h_{r,s}=\frac{(rp'-sp)^2-(p-p')^2}{4pp'}$ is a complete list of irreducible modules, and we have the following character formula:

$$\mathrm{ch}[L(c_{p/p'},h_{r,s})](q)=q^{h_{r,s}}\frac{1}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\left(q^{(n^2pp'+n(rp'-sp))}-q^{((np+r)(p'n+s))}\right).$$

2.3. $N=1$ **superconformal algebra.** The Neveu-Schwarz Lie superalgebra $SVir$, also known as $N=1$ superconformal algebra, is generated by the Virasoro modes L_n, odd generators $G_{m+\frac{1}{2}}$, $m \in \mathbb{Z}$, and the central element C; for more about this superalgebra see [9, 10]. We use central charge parametrization $c_t = \frac{3}{2} - 3(t+t^{-1}-2)$. Let $c_t \in \mathbb{C} \setminus \mathbb{Q}$, that is c_t is generic. Then we have the following result [9]:

Proposition 2.2. *Let $c = c_t$ be generic. Then $M_{ns}(c_t, h)$ is reducible for $h_{1,n} = \frac{n^2-1}{8t} - \frac{1}{4}(n-1)$. For this value of h, there is a singular vector of weight $\frac{n}{2}$ in $M_{ns}(c_t, h_n)$ that generates the maximal submodule. Consequently,*

$$\mathrm{ch}[L_{ns}(c_t, h_{1,n})](q) = q^{h_{1,n}}(-q^{1/2};q)_\infty \frac{(1-q^{\frac{n}{2}})}{(q;q)_\infty}.$$

More generally if $h_{\alpha,\beta} = \frac{1}{8t}(\alpha^2-1) - \frac{1}{4}(\alpha\beta-1) + \frac{t}{8}(\beta^2-1)$, where $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that $0 < \alpha - \beta \in 2\mathbb{Z}$ then $M_{ns}(c_t, h_{\alpha,\beta})$ is reducible and has a singular vector of degree $\frac{\alpha\beta}{2}$ that generates the maximal submodule.

If p and p' are same parity integers ≥ 2 such that $\gcd((p-p')/2, p') = 1$ then the vertex superalgebra $L_{ns}(c_{p/p'}, 0)$ is rational by a result of Adamović and the complete list of irreducible modules is given by $\{L_{ns}(c_{p/p'}, h_{r,s}) : 1 \leq r \leq p-1, 1 \leq s \leq p'-1; r \equiv s \mod 2\}$; see [1]. Moreover, we have a character formula:

$$\mathrm{ch}[L_{ns}(c_{p/p'}, h_{r,s})](q) = q^{h_{r,s}} \frac{(-q^{1/2};q)_\infty}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} \left(q^{(n^2pp'+n(rp'-sp))/2} - q^{((np+r)(p'n+s))/2} \right).$$

3. 3-fermion vertex superalgebra

In this part we are concerned with the 3-fermion vertex algebra $\mathcal{F}^{\otimes^3}$ as an extension of an Urod vertex algebras discussed in Section 1. The 2-fermion vertex algebra does not quite fit into this framework and is analyzed separately in Proposition 3.6.

It is a well-known [7] that

$$\mathcal{F}^{\otimes^3} \cong V_L \otimes \mathcal{F} \cong L_{\mathfrak{sl}_2}(2,0) \oplus L_{\mathfrak{sl}_2}(2, 2\omega_1),$$

where $L = \mathbb{Z}$ and V_L is the rank one lattice vertex superalgebra. The first isomorphism is just the Boson-Fermion correspondence. Additionally there is a superconformal vector $\tau \in \mathcal{F}^{\otimes^3}$ that combines with the conformal vector ω and together close an $N=1$ superconformal algebra of central charge $\frac{3}{2}$. With this conformal vector degree of each fermionic generator is $\frac{1}{2}$.

As we discussed in the introduction, our goal is to introduce a non-standard grading on $\mathcal{F}^{\otimes^3}$ in a way that the free fermions change their multi-grading from the usual $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \frac{1}{2} + \mathbb{N}$ to $(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) \in \mathbb{Z} + \frac{1}{2}$. What is intriguing about this new statistics is that the modified character receives only an overall shift and it equals $q^{-1/2}(-q^{1/2};q)^3_\infty$ (notice the $q^{-1/2}$ prefactor).

3.1. **New conformal structure.** In what follows we adopt Operator Product Expansion (OPE) notation. For the sake of brevity we write $a(z)$ instead of $Y(a,z)$, $a \in V$.

We denote the odd generating field of $\mathcal{F}$ by $\varphi(z) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} \varphi_n z^{-n-1/2}$ with the OPE

$$\varphi(z)\varphi(w) \sim \frac{1}{z-w}.$$

In $\mathcal{F}^{\otimes^3}$, we take a set of generators:

$$
\begin{aligned}
&\varphi_{-\frac{1}{2}}(z) = \frac{1+i\sqrt{3}}{\sqrt{6}}\left(\varphi(z)\otimes \mathbb{1}\otimes\mathbb{1}\right) + \frac{1}{\sqrt{6}}\left(\mathbb{1}\otimes\varphi(z)\otimes\mathbb{1}\right) + \frac{1-i\sqrt{3}}{\sqrt{6}}\left(\mathbb{1}\otimes\mathbb{1}\otimes\varphi(z)\right),\\
&\varphi_{\frac{1}{2}}(z) = \frac{1}{\sqrt{3}}((\varphi(z)\otimes\mathbb{1}\otimes\mathbb{1}) - (\mathbb{1}\otimes\varphi(z)\otimes\mathbb{1}) + (\mathbb{1}\otimes\mathbb{1}\otimes\varphi(z))),\\
&\varphi_{\frac{3}{2}}(z) = \frac{1-i\sqrt{3}}{\sqrt{6}}\left(\varphi(z)\otimes \mathbb{1}\otimes\mathbb{1}\right) + \frac{1}{\sqrt{6}}\left(\mathbb{1}\otimes\varphi(z)\otimes\mathbb{1}\right) + \frac{1+i\sqrt{3}}{\sqrt{6}}\left(\mathbb{1}\otimes\mathbb{1}\otimes\varphi(z)\right).
\end{aligned} \tag{1}
$$

The non-trivial OPE for these generators is given by

$$
\varphi_{-\frac{1}{2}}(z)\varphi_{\frac{3}{2}}(w) \sim \frac{2}{z-w} \qquad \text{and} \qquad \varphi_{\frac{1}{2}}(z)\varphi_{\frac{1}{2}}(w) \sim \frac{1}{z-w}. \tag{2}
$$

There is a copy of $L_{\mathfrak{sl}_2}(2,0)$ inside of $\mathcal{F}^{\otimes^3}$ with generating fields

$$
h(z) = {}^\circ_\circ\varphi_{-\frac{1}{2}}(z)\varphi_{\frac{3}{2}}(z){}^\circ_\circ, \quad e(z) = {}^\circ_\circ\varphi_{-\frac{1}{2}}(z)\varphi_{\frac{1}{2}}(z){}^\circ_\circ, \text{ and } \quad f(z) = {}^\circ_\circ\varphi_{\frac{1}{2}}(z)\varphi_{\frac{3}{2}}(z){}^\circ_\circ, \tag{3}
$$

where nontrivial OPE are given by

$$
\begin{aligned}
h(z)h(w) &\sim \frac{4}{(z-w)^2}, \qquad & e(z)f(w) &\sim \frac{2}{(z-w)^2} + \frac{h(w)}{z-w},\\
h(z)e(w) &\sim \frac{2e(w)}{z-w}, \qquad & h(z)f(w) &\sim \frac{-2f(w)}{z-w}.
\end{aligned} \tag{4}
$$

Next, we define a two-parameter family of Urod Virasoro fields in $L_{\mathfrak{sl}_2}(2,0)$ given by

$$
T_{\mathrm{Urod}}(z) = \frac{1}{4}{}^\circ_\circ e(z)f(z){}^\circ_\circ + \frac{1}{16}{}^\circ_\circ h(z)h(z){}^\circ_\circ + \frac{3}{8}\partial h(z) + \epsilon_1{}^\circ_\circ(\partial e(z))^2{}^\circ_\circ + \epsilon_1{}^\circ_\circ(\partial^2 e(z))e(z){}^\circ_\circ + \epsilon_2\partial^2 e(z), \tag{5}
$$

where ϵ_1 and ϵ_2 are an arbitrary complex parameters. Under this conformal vector, degrees of fermionic generators are:

$$
\deg(\varphi_{-1/2}) = -\frac{1}{2}, \quad \deg(\varphi_{1/2}) = \frac{1}{2}, \quad \deg(\varphi_{3/2}) = \frac{3}{2},
$$

explaining the choice of indices. We also define

$$
\begin{aligned}
G_{Urod}(z) = &-\frac{1}{4}{}^\circ_\circ f(z)\varphi_{-\frac{1}{2}}(z){}^\circ_\circ - \epsilon_2{}^\circ_\circ h(z)^2\varphi_{-\frac{1}{2}}(z){}^\circ_\circ - \epsilon_2{}^\circ_\circ(\partial h(z))\varphi_{-\frac{1}{2}}(z){}^\circ_\circ + 2(\epsilon_1 - \epsilon_2^2){}^\circ_\circ h(z)e(z)^2\varphi_{\frac{1}{2}}(z){}^\circ_\circ\\
&- 2\epsilon_2{}^\circ_\circ(\partial e(z))\varphi_{\frac{1}{2}}(z){}^\circ_\circ + 2(\epsilon_1 - 4\epsilon_2^2){}^\circ_\circ(\partial e(z))e(z)\varphi_{\frac{1}{2}}(z){}^\circ_\circ - 8\epsilon_2^3{}^\circ_\circ(\partial e(z))e(z)^2\varphi_{\frac{1}{2}}(z){}^\circ_\circ\\
&+ \frac{3}{4}{}^\circ_\circ e(z)\varphi_{\frac{3}{2}}(z){}^\circ_\circ.
\end{aligned} \tag{6}
$$

Then a straightforward computation with OPEs gives:

Proposition 3.1. *$T_{Urod}(z)$ and $G_{Urod}(z)$ close a family of $N=1$ superconformal algebras of central charge $c = -\frac{21}{2}$. Moreover, with this conformal vector, we have*

$$
\mathrm{ch}[\mathcal{F}^{\otimes^3}](q) = q^{-1/2}(-q^{1/2};q)^3_\infty.
$$

Vertex superalgebra $\mathcal{F}^{\otimes^3}$ equipped with such a superconformal structure (notice dependence on ϵ_1 and ϵ_2) is referred to as $\mathcal{U}_{N=1}$.

3.2. **Decompositions.** In this part we classify all (finite) decompositions of $\mathcal{U}_{N=1}$ as $(S)Vir_{c_1} \times (S)Vir_{c_2}$-modules, with the overall central charge $c_1 + c_2 = -\frac{21}{2}$ and with c_i from minimal series. We first have to narrow down possible values of central charges.

Proposition 3.2. *(a) Let c_1 and c_2 be from the minimal series for $SVir$ and $c_1c_2 \neq 0$. If $L_{ns}(c_1,0) \otimes L_{ns}(c_2,0)$ conformally embeds into $\mathcal{U}_{N=1}$, then $c_1 = c_2 = -\frac{21}{4}$. This is also an embedding of $N=1$ vertex superalgebras.*

(b) Let c_1 and c_2 be from the minimal series for Vir and $SVir$, respectively, and $c_1 \neq 0$. If $L(c_1,0) \otimes L_{ns}(c_2,0)$ conformally embeds into $\mathcal{U}_{N=1}$, then $(c_1,c_2) = (-\frac{68}{7}, -\frac{11}{14})$, $(c_1,c_2) = (-\frac{21}{4}, -\frac{21}{4})$, or $(c_1,c_2) = (\frac{1}{2}, -11)$.

(c) Let c_1 and c_2 be among minimal series for Vir. If $L(c_1,0) \otimes L(c_2,0)$ conformally embeds into $\mathcal{U}_{N=1}$, then $c_1 = c_2 = -\frac{21}{4}$ or $(c_1,c_2) = (-\frac{68}{7}, -\frac{11}{14})$.

Proof. To see that these cases include all possible examples, we first setup equations for central charges and asymptotic dimensions. For part (a) using e.g. [13], we get a relation among central charges to be

$$-\frac{21}{2} = \underbrace{\frac{3}{2}\left(1 - 2\frac{(p-p')^2}{pq}\right)}_{=c_1} + \underbrace{\frac{3}{2}\left(1 - 2\frac{(r-s)^2}{rs}\right)}_{=c_2}$$

and for asymptotic dimensions (here we are using that Weber's function $\mathfrak{f}$ has asymptotic dimension $\frac{1}{2}$):

$$\frac{3}{2} = \frac{3}{2}\left(1 - \frac{8}{pp'}\right) + \frac{3}{2}\left(1 - \frac{8}{rs}\right).$$

Solving this subject to $\gcd((p-p')/2, p') = \gcd((r-s)/2, r) = 1$ give the only solution $c_1 = c_2 = -\frac{21}{4}$. Similarly we see parts (b) and (c). $\square$

We will now explicitly construct all the vertex algebra embeddings foreseen in Proposition 3.2.

3.3. **Superconformal embedding** $(c_1, c_2) = (-\frac{21}{4}, -\frac{21}{4})$**.** We first construct a (super)conformal embedding identified in Proposition 3.2 (a). We start $(c_1,c_2) = (-\frac{21}{4}, -\frac{21}{4})$. A straightforward but tedious computation using OPE reveals a 2-parameter family of commuting copies of $SVir$ of central charge $c = -\frac{21}{4}$ given by:

$$\begin{aligned} T^{(1)}_{3|8}(z) =& \frac{1}{16\epsilon} f(z) + \eta {}^\circ_\circ e(z) f(z) {}^\circ_\circ - \frac{3-16\eta}{32} {}^\circ_\circ h(z)^2 {}^\circ_\circ - \frac{5-16\eta}{16} \partial h(z) - 3(4\eta^2 - \eta)\epsilon {}^\circ_\circ h(z)^2 e(z) {}^\circ_\circ \\ &+ \frac{128\eta^2 - 32\eta - 3}{8} \epsilon {}^\circ_\circ h(z) \partial e(z) {}^\circ_\circ - 2(4\eta^2 - \eta)\epsilon {}^\circ_\circ (\partial h(z)) e(z) {}^\circ_\circ - \frac{32\eta + 1}{8} \epsilon \partial^2 e(z) \\ &+ \frac{3(512\eta^3 - 192\eta^2 - 8\eta + 2)}{8} \epsilon^2 {}^\circ_\circ (\partial h(z)) e(z)^2 {}^\circ_\circ - 4\eta(32\eta^2 - 28\eta - 1)\epsilon^2 {}^\circ_\circ (\partial^2 e(z)) e(z) {}^\circ_\circ \\ &+ \frac{3(256\eta^2 - 64\eta + 21)}{8} \epsilon^3 {}^\circ_\circ (\partial^2 e(z)) e(z)^2 {}^\circ_\circ, \end{aligned} \tag{7}$$

$$
\begin{aligned}
G^{(1)}_{3|8}(z) =& \frac{1}{8\epsilon}\varphi_{\frac{3}{2}}(z) + \frac{8\eta+1}{4} {}^\circ_\circ e(z)\varphi_{\frac{3}{2}}(z)^\circ_\circ + 2\eta^\circ_\circ h(z)\varphi_{\frac{1}{2}}(z)^\circ_\circ + (16\eta^2-1)\epsilon^\circ_\circ(\partial h(z))\varphi_{-\frac{1}{2}}(z)^\circ_\circ \\
&+ 2(8\eta^2+2\eta-1)\epsilon^\circ_\circ h(z)e(z)\varphi_{\frac{1}{2}}(z)^\circ_\circ + \frac{16\eta-1}{2}\epsilon^\circ_\circ(\partial e(z))\varphi_{\frac{1}{2}}(z)^\circ_\circ \\
&- 4(16\eta^2-2\eta+1)\epsilon^2{}^\circ_\circ h(z)e(z)^2\varphi_{\frac{1}{2}}(z)^\circ_\circ - (128\eta^2-40\eta-1)\epsilon^2{}^\circ_\circ(\partial e(z))e(z)\varphi_{\frac{1}{2}}(z)^\circ_\circ \\
&- \frac{4096\eta^3-1536\eta^2+192\eta-25}{2}\epsilon^3{}^\circ_\circ(\partial e(z))e(z)^2\varphi_{\frac{1}{2}}(z)^\circ_\circ
\end{aligned} \tag{8}
$$

and

(9)

$$
\begin{aligned}
T^{(2)}_{3|8}(z) =& -\frac{1}{16\epsilon}f - \frac{4\eta-1}{4} {}^\circ_\circ e(z)f(z)^\circ_\circ + \frac{16\eta-1}{32} {}^\circ_\circ h(z)^2{}^\circ_\circ + \frac{16\eta+1}{16}\partial h(z) + (12\eta^2-3\eta)\epsilon^\circ_\circ h(z)^2 e(z)^\circ_\circ \\
&- \frac{128\eta^2-32\eta+3}{8}\epsilon^\circ_\circ h(z)\partial e(z)^\circ_\circ + (8\eta^2-2\eta)\epsilon^\circ_\circ(\partial h(z))e(z)^\circ_\circ - \frac{32\eta-9}{8}\epsilon\partial^2 e(z) \\
&+ \frac{3}{8}\left(-512\eta^3+192\eta^2+8\eta-3\right)\epsilon^2{}^\circ_\circ(\partial h(z))e(z)^2{}^\circ_\circ + (128\eta^3+16\eta^2-36\eta+6)\epsilon^2{}^\circ_\circ(\partial^2 e(z))e(z)^\circ_\circ \\
&- \frac{3}{8}\left(256\eta^2-64\eta+21\right)\epsilon^3{}^\circ_\circ(\partial^2 e(z))e(z)^2{}^\circ_\circ,
\end{aligned}
$$

(10)

$$
\begin{aligned}
G^{(2)}_{3|8}(z) =& -\frac{1}{8\epsilon}\varphi_{\frac{3}{2}}(z) - \frac{8\eta-3}{4} {}^\circ_\circ e(z)\varphi_{\frac{3}{2}}(z)^\circ_\circ - \frac{4\eta-1}{2} {}^\circ_\circ h(z)\varphi_{\frac{1}{2}}(z)^\circ_\circ - 8(2\eta^2-1)\epsilon^\circ_\circ(\partial h(z))\varphi_{-\frac{1}{2}}(z)^\circ_\circ \\
&- 4(4\eta^2-3\eta)\epsilon^\circ_\circ h(z)e(z)\varphi_{\frac{1}{2}}(z)^\circ_\circ - \frac{16\eta-3}{2}\epsilon^\circ_\circ(\partial e(z))\varphi_{\frac{1}{2}}(z)^\circ_\circ \\
&+ 2(32\eta^2-12\eta+3)\epsilon^2{}^\circ_\circ h(z)e(z)^2\varphi_{\frac{1}{2}}(z)^\circ_\circ - (128\eta^2-24\eta-3)\epsilon^2{}^\circ_\circ(\partial e(z))e(z)\varphi_{\frac{1}{2}}(z)^\circ_\circ \\
&+ \frac{4096\eta^3-1536\eta^2+192\eta+9}{2}\epsilon^3{}^\circ_\circ(\partial e(z))e(z)^2\varphi_{\frac{1}{2}}(z)^\circ_\circ.
\end{aligned}
$$

Here we have

$$
\begin{aligned}
T^{(1)}_{3|8}(z) + T^{(2)}_{3|8}(z) &= T_{Urod}\Big|_{\substack{\epsilon_1\to 2(64\eta^2-16\eta+3)\epsilon^2 \\ \epsilon_2\to(1-8\eta)\epsilon}} \\
G^{(1)}_{3|8}(z) + G^{(2)}_{3|8}(z) &= G_{Urod}\Big|_{\substack{\epsilon_1\to 2(64\eta^2-16\eta+3)\epsilon^2 \\ \epsilon_2\to(1-8\eta)\epsilon}}.
\end{aligned} \tag{11}
$$

Theorem 3.3. *As $N=1$ vertex operator superalgebras, we have*

$$
\mathcal{U}_{N=1} = L_{ns}\left(-\frac{21}{4},0\right)^{\otimes^2} \oplus L_{ns}\left(-\frac{21}{4},-\frac{1}{4}\right)^{\otimes^2}.
$$

Proof. From the discussion above and relations (11) we know that there are two commuting copies of the $N=1$ superconformal algebra of central charge $-\frac{21}{4}$. So we only have to rule out that one of the copies is $V_{ns}(-\frac{21}{4},0)$. But this easily follows from the previous observation of the asymptotic growth of characters involved. The rest follows using the rationality of $L_{ns}(-\frac{21}{4},0)$ [1] and known character formulas for irreducible modules (see Section 2). □

Comparing characters on both sides of Theorem 3.3, after multiplying with $q^{\frac{1}{2}}$, gives the following identity which can be also proven directly using q-series.

Corollary 3.4.

$$(-q^{1/2};q)^3_\infty = \frac{1}{\prod_{n\geq 1; n\not\equiv 0,\pm 2,\pm 3 \pmod 8}(1-q^{n/2})^2} + q^{\frac{1}{2}}\frac{1}{\prod_{n\geq 1; n\not\equiv 0,\pm 1,\pm 2 \pmod 8}(1-q^{n/2})^2}.$$

3.4. **(Super)conformal embedding** $(c_1,c_2)=(-11,\frac{1}{2})$**.** There is a Virasoro field with central charge $\frac{1}{2}$ given by

$$\begin{aligned} T_{3|4}(z) =& \frac{1}{4}{}^{\circ}_{\circ}e(z)f(z){}^{\circ}_{\circ} - \frac{1}{16}{}^{\circ}_{\circ}h(z)^2{}^{\circ}_{\circ} - \frac{1}{8}\partial h(z) + \epsilon{}^{\circ}_{\circ}h(z)^2e(z){}^{\circ}_{\circ} - \epsilon{}^{\circ}_{\circ}h(z)\partial e(z){}^{\circ}_{\circ} - \epsilon{}^{\circ}_{\circ}(\partial h(z))e(z){}^{\circ}_{\circ} \\ & - 4\epsilon^2{}^{\circ}_{\circ}(\partial e(z))^2{}^{\circ}_{\circ} + 6\epsilon^2{}^{\circ}_{\circ}(\partial h(z))e(z)^2{}^{\circ}_{\circ} - 4\epsilon^2{}^{\circ}_{\circ}(\partial^2 e(z))e(z){}^{\circ}_{\circ}, \end{aligned} \tag{12}$$

and a commuting copy of the $N=1$ superconformal algebra with

$$\begin{aligned} T(z) =& \frac{1}{8}{}^{\circ}_{\circ}h(z)^2{}^{\circ}_{\circ} + \frac{1}{2}\partial h(z) + \epsilon{}^{\circ}_{\circ}h(z)^2e(z){}^{\circ}_{\circ} - \epsilon{}^{\circ}_{\circ}h(z)\partial e(z){}^{\circ}_{\circ} + \epsilon{}^{\circ}_{\circ}(\partial h(z))e(z){}^{\circ}_{\circ} - 2\epsilon\partial^2 e(z) \\ & - 6\epsilon^2{}^{\circ}_{\circ}(\partial h(z))e(z)^2{}^{\circ}_{\circ} + \eta{}^{\circ}_{\circ}(\partial e(z))^2{}^{\circ}_{\circ} + \eta{}^{\circ}_{\circ}(\partial^2 e(z))e(z){}^{\circ}_{\circ}, \end{aligned} \tag{13}$$

and

(14)
$$\begin{aligned} G(z) =& -\frac{1}{2\sqrt{\eta-12\epsilon^2}}\varphi_{\frac{3}{2}}(z) - \frac{2\epsilon}{\sqrt{\eta-12\epsilon^2}}{}^{\circ}_{\circ}e(z)\varphi_{\frac{3}{2}}(z){}^{\circ}_{\circ} - \frac{2\epsilon}{\sqrt{\eta-12\epsilon^2}}{}^{\circ}_{\circ}h(z)\varphi_{\frac{1}{2}}(z){}^{\circ}_{\circ} \\ & - \frac{2(18\epsilon^2-\eta)}{\sqrt{9\eta-108\epsilon^2}}{}^{\circ}_{\circ}(\partial h(z))\varphi_{-\frac{1}{2}}(z){}^{\circ}_{\circ} - \frac{108\epsilon^2-7\eta}{6\sqrt{\eta-12\epsilon^2}}{}^{\circ}_{\circ}h(z)e(z)\varphi_{\frac{1}{2}}(z){}^{\circ}_{\circ} + \frac{4(\epsilon^3-\epsilon\eta)}{\sqrt{\eta-12\epsilon^2}}{}^{\circ}_{\circ}(\partial e(z))e(z)\varphi_{\frac{1}{2}}(z){}^{\circ}_{\circ} \\ & - \frac{720\epsilon^4-216\epsilon^2\eta+13\eta^2}{6\sqrt{\eta-12\epsilon^2}}{}^{\circ}_{\circ}(\partial e(z))^2\varphi_{-\frac{1}{2}}(z){}^{\circ}_{\circ}. \end{aligned}$$

Here we have

$$T_{3|4}(z) + T(z) = T_{Urod}(z)\Big|_{\substack{\epsilon_1\to -4\epsilon^2+\eta \\ \epsilon_2\to -2\epsilon}}.$$

Then using the same argument as in the proof of Theorem 3.3.

Proposition 3.5. *As an* $L_{ns}(-11,0)\otimes L\left(\frac{1}{2},0\right)$*-module*

$$\begin{aligned} \mathcal{U}_{N=1} =& L_{ns}(-11,0)\otimes L\left(\frac{1}{2},0\right) \oplus L_{ns}(-11,0)\otimes L\left(\frac{1}{2},\frac{1}{2}\right) \\ & \oplus L_{ns}\left(-11,-\frac{1}{2}\right)\otimes L\left(\frac{1}{2},0\right) \oplus L_{ns}\left(-11,-\frac{1}{2}\right)\otimes L\left(\frac{1}{2},\frac{1}{2}\right). \end{aligned}$$

Using $L\left(\frac{1}{2},0\right)\oplus L\left(\frac{1}{2},\frac{1}{2}\right) = \langle \varphi_{\frac{1}{2}}(z)\rangle = \mathcal{F}$ we get the commutant

$$Com(\mathcal{F},\mathcal{F}^{\otimes^3}) = L_{ns}(-11,0)\oplus L_{ns}(-11,-\frac{1}{2})$$

and therefore

$$\mathcal{U}_{N=1} = \left(L_{ns}(-11,0)\oplus L_{ns}(-11,-\frac{1}{2})\right)\otimes\mathcal{F}.$$

As a conclusion we observe:

Proposition 3.6. *The two fermion vertex superalgebra $\mathcal{F}^{\otimes 2}$ admits an $N=1$ conformal structure of central charge $c=-11$ such that*

$$\mathrm{ch}[\mathcal{F}^{\otimes 2}](q) = q^{-1/2}(-q^{1/2};q)^2_\infty.$$

Explicitly, inside of $\mathcal{F}^{\otimes 2} = \left\langle \varphi_{-\frac{1}{2}}(z), \varphi_{\frac{3}{2}}(z)\right\rangle$, we have

$$T(z) = \frac{1}{4} {}^\circ_\circ\varphi_{-\frac{1}{2}}(z)\partial\varphi_{\frac{3}{2}}(z){}^\circ_\circ + \frac{3}{2}{}^\circ_\circ(\partial\varphi_{-\frac{1}{2}}(z))\varphi_{\frac{3}{2}}(z){}^\circ_\circ - \frac{\epsilon^2}{2}{}^\circ_\circ(\partial^2\varphi_{-\frac{1}{2}}(z))\partial\varphi_{-\frac{1}{2}}(z){}^\circ_\circ - \frac{\epsilon^2}{2}{}^\circ_\circ(\partial^3\varphi_{-\frac{1}{2}}(z))\varphi_{-\frac{1}{2}}(z){}^\circ_\circ$$

and

$$G(z) = -\frac{1}{2\epsilon}\varphi_{\frac{3}{2}}(z) + \frac{\epsilon}{2}{}^\circ_\circ(\partial\varphi_{-\frac{1}{2}})\varphi_{-\frac{1}{2}}(z)\varphi_{\frac{3}{2}}(z){}^\circ_\circ + \frac{13\epsilon^3}{12}{}^\circ_\circ(\partial^2\varphi_{-\frac{1}{2}})(\partial\varphi_{-\frac{1}{2}}(z))\varphi_{-\frac{1}{2}}(z){}^\circ_\circ + \frac{11\epsilon}{6}\partial^2\varphi_{-\frac{1}{2}}(z).$$

3.5. **(Super)conformal embedding $(c_1,c_2) = (-\frac{11}{14}, \frac{-68}{7})$.** Next result will be proven in a more conceptual way later on so now we only state the result and observe that we can again use Theorem 3.3.

Proposition 3.7. *As an $SVir_{-11/14}\times Vir_{-68/7}$-module:*

$$\begin{aligned}\mathcal{U}_{N=1} = {} & L_{ns}(-11/14,0)\otimes L(-68/7,0)\oplus \\ & \oplus L_{ns}(-11/14,-1/14)\otimes L(-68/7,-3/7)\oplus L_{ns}(-11/14,2/7)\otimes L(-68/7,-2/7).\end{aligned}$$

Since $-\frac{11}{14}$ is from the minimal series of central charge for the Virasoro algebra, we also get $Vir_{-11/14}\times Vir_{-68/7}$ decomposition (this decomposition was also mentioned in [4]) of $\mathcal{U}_{N=1}$ using

$$\begin{aligned}L_{ns}(-11/14,0) &= L(-11/14,0)\oplus L(-11/4,3/2)\oplus L(-11/14,4)\oplus L(-11/14,25/2)\\ L_{ns}(-11/14,-1/14) &= L(-11/14,-1/14)\oplus L(-11/4,3/7)\oplus L(-11/14,17/7)\oplus L(-11/14,69/14)\\ L_{ns}(-11/14,2/7) &= L(-11/14,2/7)\oplus L(-11/4,11/14)\oplus L(-11/14,25/14)\oplus L(-11/14,58/7).\end{aligned}$$

Next we give explicit conformal and superconformal vectors in this case. The Virasoro field with central charge $-\frac{68}{7}$ is given by

$$\begin{aligned}T_{2|7}(z) = {} & -\frac{1}{14\epsilon}f(z) + \frac{1-4\eta}{4}{}^\circ_\circ e(z)f(z){}^\circ_\circ + \frac{28\eta-3}{56}{}^\circ_\circ h(z)^2{}^\circ_\circ + \frac{28\eta+3}{28}\partial h(z)\\ & + \frac{1}{32}(4\eta-1)(84\eta-5)\epsilon{}^\circ_\circ h(z)^2e(z){}^\circ_\circ + \frac{1}{16}(4\eta-1)(28\eta+5)\epsilon{}^\circ_\circ(\partial h(z))e(z){}^\circ_\circ\\ & - \frac{1}{56}(28\eta-11)(28\eta-1)\epsilon{}^\circ_\circ h(z)\partial e(z){}^\circ_\circ + -\frac{3}{14}(28\eta-11)\epsilon\partial^2 e(z)\\ & + \frac{2}{343}\left(21952\eta^3+8624\eta^2-15148\eta+2137\right)\epsilon^2{}^\circ_\circ(\partial e(z))^2{}^\circ_\circ\\ & + \frac{2}{343}\left(21952\eta^3+8624\eta^2-13804\eta+1609\right)\epsilon^2{}^\circ_\circ(\partial^2 e(z))e(z){}^\circ_\circ\\ & + -\frac{3}{343}(28\eta-11)(28\eta-3)(28\eta+1)\epsilon^2{}^\circ_\circ(\partial h(z))e(z)^2{}^\circ_\circ\\ & + -\frac{384}{2401}\left(784\eta^2-616\eta+69\right)\epsilon^3{}^\circ_\circ(\partial^2 e(z))e(z)^2{}^\circ_\circ,\end{aligned} \tag{15}$$

and the $N=1$ superconformal algebra with central charge $-\frac{11}{14}$ is generated by

$$
\begin{aligned}
T_{7|12}(z) =& \frac{1}{14\epsilon} f(z) + \eta {}^\circ_\circ e(z) f(z) {}^\circ_\circ + \frac{1}{112}(13-56\eta) {}^\circ_\circ h(z)^2 {}^\circ_\circ + \frac{1}{56}(15-56\eta)\partial h(z) \\
& - \frac{1}{28}(4\eta-1)(84\eta-5)\epsilon {}^\circ_\circ h(z)^2 e(z) {}^\circ_\circ - \frac{1}{14}(4\eta-1)(28\eta+5)\epsilon {}^\circ_\circ (\partial h(z)) e(z) {}^\circ_\circ \\
& + \frac{1}{49}(28\eta-11)(28\eta-1)\epsilon {}^\circ_\circ h(z)\partial e(z) {}^\circ_\circ - \frac{2}{49}(28\eta-11)\epsilon\partial^2 e(z) \\
& - \frac{2}{343}(28\eta+1)\left(784\eta^2-504\eta+93\right)\epsilon^2 {}^\circ_\circ (\partial e(z))^2 {}^\circ_\circ \\
& - \frac{2}{343}\left(21952\eta^3-13328\eta^2+3444\eta-435\right)\epsilon^2\epsilon^2 {}^\circ_\circ (\partial^2 e(z)) e(z) {}^\circ_\circ \\
& + \frac{3}{343}(28\eta-11)(28\eta-3)(28\eta+1)\epsilon^2\epsilon^2 {}^\circ_\circ (\partial h(z)) e(z)^2 {}^\circ_\circ \\
& + \frac{384}{2401}\left(784\eta^2-616\eta+69\right)\epsilon^3 {}^\circ_\circ (\partial^2 e(z)) e(z)^2 {}^\circ_\circ ,
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
G_{7|12}(z) =& \frac{1}{2\epsilon\sqrt{21}}\varphi_{\frac{3}{2}} - \frac{28\eta+1}{4\sqrt{21}} {}^\circ_\circ f(z)\varphi_{-\frac{1}{2}}(z) {}^\circ_\circ - \frac{1}{4}\sqrt{\frac{7}{3}}(4\eta+1) {}^\circ_\circ h(z)\varphi_{\frac{1}{2}}(z) {}^\circ_\circ \\
& + \frac{1}{16}\sqrt{\frac{7}{3}}(4\eta-1)(28\eta+9)\epsilon {}^\circ_\circ h(z)e(z)\varphi_{\frac{1}{2}}(z) {}^\circ_\circ + \frac{1}{8}\sqrt{\frac{3}{7}}(28\eta-3)^2\epsilon^2 {}^\circ_\circ h(z)e(z)^2\varphi_{\frac{1}{2}}(z) {}^\circ_\circ \\
& + \frac{1}{16}\sqrt{\frac{7}{3}}(4\eta-1)(28\eta-3)\epsilon {}^\circ_\circ (\partial h(z))\varphi_{-\frac{1}{2}}(z) {}^\circ_\circ + \frac{1}{2}\sqrt{\frac{3}{7}}(28\eta-3)\epsilon {}^\circ_\circ (\partial e(z))\varphi_{\frac{1}{2}}(z) {}^\circ_\circ \\
& - \frac{1}{4}\sqrt{\frac{3}{7}}\left(784\eta^2-280\eta+37\right)\epsilon^2 {}^\circ_\circ (\partial e(z))e(z)\varphi_{\frac{1}{2}}(z) {}^\circ_\circ \\
& + \frac{1}{8}\sqrt{21}\left(3136\eta^3-1456\eta^2+332\eta-25\right)\epsilon^3 {}^\circ_\circ (\partial e(z))e(z)^2\varphi_{\frac{1}{2}}(z) {}^\circ_\circ
\end{aligned} \tag{17}
$$

Here we note that

$$
T_{2|7}(z) + T_{7|12}(z) = T_{Urod}(z)\Big|_{\substack{\epsilon_1 \to \frac{1}{8}\left(784\eta^2-616\eta+73\right)\epsilon^2 \\ \epsilon_2 \to -\frac{1}{4}(28\eta-11)\epsilon}}.
$$

4. Level two Urod algebra and decompositions

4.1. **Drinfeld-Sokolov reduction.** In this part we closely follow [3] and [4]. For $k \neq -2$, let H_{DS} denotes the Drinfeld-Sokolov reduction functor from the catetegory of $V_{\mathfrak{sl}_2}(k,0)$-modules to the category of $V(c_k,0)$-modules where $c_k = 1 - \frac{6(k+1)^2}{k+2}$. Also, denote by H^{Δ}_{DS} the diagonal Drinfeld-Sokolov reduction as discussed in [3, 4].

First we assume that k is generic. Then $L_{\mathfrak{sl}_2}(k, n\omega_1)$, $n \geq 0$, are irreducible $L_{\mathfrak{sl}_2}(k,0)$-modules, so called Weyl modules. Then

$$
H^0_{DS}(L_{\mathfrak{sl}_2}(k,n\omega_1)) = L(c_k, h_{1,n+1})
$$

and $H^i_{DS}(L_{\mathfrak{sl}_2}(k,n\omega_1)) = 0$, $i \geq 1$.

For admissible levels $k \in -2 + \mathbb{Q}_{>0}$ [12], we have

$$H^0_{DS}\left(L_{\mathfrak{sl}_2}\left(-2+\frac{p}{p'}, ((m-1)-(k+2)(m'-1))\omega_1\right)\right) = L(c_{p/p'}, h_{m,m'}),$$

where as before $0 < m < p$ and $0 < m' < p'$.

In [4, 3] a result on decomposition of the $\ell = 1$ level Urod algebra $\mathcal{U} = L_{\mathfrak{sl}_2}(1,0)$, of central charge $c = -5$, tensored with a generic Virasoro algebra $V(c,0)$ is given. In other words, the vertex algebra

$$V(c_t, 0) \otimes \mathcal{U}$$

is decomposed as a $V(c_{1+t}, 0) \otimes V(c_{1+1/t}, 0)$-module. This requires a decomposition of the sum of two conformal vectors:

$$T_{c_t} + T_{Urod} = T_{c_{1+t}} + T_{c_{1+1/t}}$$

where $T_{c_{1+t}}$ and $T_{c_{1+1/t}}$ commute with each other. Explicit formulas for these conformal vectors appearing on the right-hand side were presented in loc.cit.

A generalization and conceptual explanation of this result for any semi-simple Lie algebra $\mathfrak{g}$ is given in [3] among other things. In particular, if $\mathfrak{g}$ is simply laced, then the commuting pair of Virasoro algebra is replaced with a commuting pair of principal affine simple W-algebra $\mathcal{W}_k(\mathfrak{g}) \otimes \mathcal{W}_\ell(\mathfrak{g})$ for certain k and ℓ; for a precise statement [3, Section 8].

4.2. **Adding a super structure.** Our first result is essentially from [3]:

Theorem 4.1 (ACF). *Let $\mathcal{U}_{N=1}$ be equipped with a conformal Urod structure with $\ell = 2$ for $\mathfrak{sl}_2$ as in [3], then we have an isomorphism of vertex superalgebras* [3]

$$H^{\Delta}_{DS}(L_{\mathfrak{sl}_2}(k,0) \otimes \mathcal{U}_{N=1}) = L(c_k, 0) \otimes \mathcal{U}_{N=1}.$$

Moreover, for any $L_{\mathfrak{sl}_2}(k,0)$-module M, we have

$$H^{\Delta}_{DS}(M \otimes \mathcal{U}_{N=1}) = H^0_{DS}(M) \otimes \mathcal{U}_{N=1}.$$

Proof. We only have to observe that $\mathcal{U}_{N=1} = L_{sl_2}(2,0) \oplus L_{sl_2}(2, 2\omega_1)$ has a vertex superalgebra structure and that H_{DS} commutes with direct sums. □

Next we exhibit two special cases: generic and admissible level k.

4.3. **Generic case.** Let $L(c_t, 0)$ be a generic vertex algebra then we have the following decomposition in parallel with [4].

Theorem 4.2. *Let $L(c_t, 0)$ be generic and $\mathcal{U}_{N=1}$ equipped with the Urod structure as in [3]. Then we have a super conformal embedding*

$$L(c_{2+t}, 0) \otimes L_{ns}(c_{1+\frac{2}{t}}, 0) \hookrightarrow L(c_t, 0) \otimes \mathcal{U}_{N=1}. \tag{18}$$

Moreover, we have a character identity:

$$\mathrm{ch}[L(c_t,0) \otimes \mathcal{U}_{N=1}](q) = \sum_{\substack{n \geq 1 \\ n \equiv 1 \mod 2}} \mathrm{ch}[L(c_{2+t}, h_{1,n})](q) \cdot \mathrm{ch}[L_{ns}(c_{1+\frac{2}{t}}, h_{1,n})](q).$$

[3] This is not an isomorphism of $N = 1$ vertex superalgebras as superconformal vectors on two sides are different.

Proof. We first require a decomposition of

$$T^{Vir}_{c_t} + T_{Urod} = T^{Vir}_{c_{2+t}} + T^{NS}_{c_{1+2/t}},$$

inside the tensor product of $L(c_t,0)\otimes\mathcal{U}_{N=1}$ where $T^{NS}_{c_{1+2/t}}$ is a conformal vector that combines into the $N=1$ superconformal algebra with the same central charge. This was achieved using explicit computations in Section 9. Additionally, the main result of Section 5 shows that a specialization of the Urod family of conformal vectors of $\mathcal{U}_{N=1}$ constructed earlier coincides with the level 2 Urod conformal vector constructed in [3]; see our formula (23).

For the character identity we only have to recall Propositions 2.1 and 2.2, and an easy q-series identity

$$\sum_{n\geq 0} q^{n^2/2-n}\left(1-q^{(2n+1)/2})(1-q^{2n+1})\right) = (1-q)(-q^{1/2};q)^2_\infty (q;q)_\infty,$$

which follows directly from the Jacobi triple product identity $(-q^{1/2};q)^2_\infty(q;q)_\infty = \sum_{n\in\mathbb{Z}} q^{n^2/2}$. $\square$

Remark 4.3. *Theorem 4.2 clearly suggests a decomposition*

$$L(c_t,0)\otimes\mathcal{U}_{N=1} = \sum_{\substack{n\geq 1\\ n\equiv 1 \mod 2}} L(c_{2+t},h_{1,n})\otimes L_{ns}(c_{1+\frac{2}{t}},h_{1,n}).$$

This follows using Theorem 4.1 together with an isomorphism of $SVir\times\widehat{\mathfrak{sl}_2}$-modules:

$$L_{\mathfrak{sl}_2}(k,0)\otimes\mathcal{U}_{N=1} = \bigoplus_{n\equiv 1 \mod 2} L_{ns}(c_{1+2/t},h_{1,n})\otimes L_{\mathfrak{sl}_2}(k+2,n\omega_1), \tag{19}$$

However, (19) is somewhat difficult to prove just by relying on the methods discussed in this paper. In addition to semisimplicity of the relevant category of $V_{\mathfrak{sl}_2}(k,0)$-modules (which is known), we also require a delicate argument that each $L_{\mathfrak{sl}_2}(k+2,n\omega_1)$ appears in the decomposition with the required q-multiplicity. Finally, we require a q-series identity:

$$\sum_{n\geq 0} q^{n^2/2}(x^{-n}+\cdots+1+\cdots+x^n)\left(1-q^{(2n+1)/2}\right) = (q;q)_\infty(-q^{1/2}x;q)_\infty(-q^{1/2}x^{-1};q)_\infty,$$

another easy consequence of the (x,q)-Jacobi triple product identity

$$(q;q)_\infty(-q^{1/2}x;q)_\infty(-q^{1/2}x^{-1};q)_\infty = \sum_{n\in\mathbb{Z}} x^n q^{n^2/2}.$$

4.4. **Admissible level.** For admissible levels we have:

Theorem 4.4. *Let (p,p') be such that $\gcd(p,p')=1$ and $p,p'\geq 2$. Then we have the following decomposition as a $Vir_{c_{p',2p'+p}}\times SVir_{p,2p'+p}$-module:*

$$L(c_{p/p'},0)\otimes\mathcal{U}_{N=1} = \sum_{\substack{n\geq 1\\ n\equiv 1 \mod 2}}^{2p'+p-1} L(c_{p'/(2p'+p)},h_{1,n})\otimes L_{ns}(c_{p/(2p'+p)},h_{1,n}).$$

Proof. This follows from $H^{\Delta}_{DS}(L_{sl_2}(-k+\frac{p}{p'},0)\otimes\mathcal{U}_{N=1})\cong H_{DS}(L_{sl_2}(-k+\frac{p}{p'},0))\otimes\mathcal{U}_{N=1}$ combined with the decomposition formula:

$$L_{sl_2}(-2+\frac{p}{p'},0)\otimes(L_{sl_2}(2,0)\oplus L_{sl_2}(2,\omega_1)) = \bigoplus_{\substack{0<i<2p'+p\\ i=1 \mod 2}} L_{ns}(c_{p/(2p'+p)},h_{1,i})\otimes L_{\mathfrak{sl}_2}(\frac{p}{p'},(i-1)\omega_1),$$

and more generally,

$$
\begin{aligned}
&L_{\mathfrak{sl}_2}\left(-2+\frac{p}{p'},((m-1)-(k+2)(m'-1))\omega_1\right)\otimes(L_{sl_2}(2,0)\oplus L_{sl_2}(2\Lambda_1))\\
&=\bigoplus_{\substack{0<i<2p'+p\\ i=m' \mod 2}} L_{ns}(c_{p/(2p'+p)},h_{m,i})\otimes L_{\mathfrak{sl}_2}\left(\frac{p}{p'},(i-1-(k+2)(m'-1))\omega_1\right)
\end{aligned}
$$

given in [13] (see also [1]). An application of Theorem 4.1 now gives

(20)

$$
\begin{aligned}
L(c_t,h_{m,m'})\otimes\mathcal{U}_{N=1} &= \bigoplus_{\substack{0<i<2p'+p\\ i=m' \mod 2}} L_{ns}(c_{p/(2p'+p)},h_{m,i})\otimes H^0_{DS}(L_{\mathfrak{sl}_2}(\frac{p}{p'},(i-1-(k+2)(m'-1))\omega_1))\\
&= \bigoplus_{\substack{0<i<2p'+p\\ i=m' \mod 2}} L_{ns}(c_{p/(2p'+p)},h_{m,i})\otimes L(c_{(2p'+p)/p'},h_{i,m'}).
\end{aligned}
$$

Setting $(m,m')=(1,1)$ now gives the formula. □

4.5. **Super extensions.** The present setup does not give rise to decomposition formulas for the tensor product $L_{ns}(c_{r,s},0)\otimes\mathcal{U}_{N=1}$. However, when we have conformal embedding $L(c_{p,p'},0)\hookrightarrow L_{ns}(c_{r,s},0)$; in other words when the central charge belongs to both minimal series for Vir and $SVir$ we can get such decomposition using Theorem 4.4 and formula (20). There are precisely three such embeddings with central charges: $c_{p,p'}=\frac{7}{10},-\frac{21}{4}$, and $-\frac{11}{14}$ (there is also $c_{p,p'}=0$ but then the simple vertex algebra is trivial).

We start from $c=c_{4,5}=\frac{7}{10}$. Then $L_{ns}(\frac{7}{10},0)=L(c_{4,5},h_{1,1})\oplus L(c_{4,5},h_{1,4})$. Using the formulas above we get

$$
L(c_{4,5},h_{1,1})\otimes\mathcal{U}_{N=1}=\bigoplus_{i\equiv 1 \mod 2}^{13} L_{ns}(c_{4,14},h_{1,i})\otimes L(c_{5,14},h_{1,i}),
$$

$$
L(c_{4,5},h_{1,4})\otimes\mathcal{U}_{N=1}=\bigoplus_{i\equiv 1 \mod 2}^{13} L_{ns}(c_{4,14},h_{1,i})\otimes L(c_{5,14},h_{4,i}).
$$

Combining these isomorphisms gives

$$
L_{ns}(\frac{7}{10},0)\otimes\mathcal{U}_{N=1}=\bigoplus_{i\equiv 1 \mod 2}^{13} L_{ns}(c_{4,14},h_{1,i})\otimes(L(c_{5,14},h_{1,i})\oplus L(c_{5,14},h_{4,i})),
$$

and therefore the coset superalgebra of $L_{ns}(c_{4,14},0)$ is vertex superalgebra $L(c_{5,14},h_{1,1})\oplus L(c_{5,14},h_{4,1})$, of type $(2,9)$.

Next example is $c=c_{3,8}=-\frac{21}{4}$, where $L_{ns}(-\frac{21}{4},0)=L(c_{3,8},0)\oplus L(c_{3,8},h_{1,7})$. Similar computation as above gives an isomorphism:

$$
L_{ns}(-\frac{21}{4},0)\otimes\mathcal{U}_{N=1}=\sum_{i\equiv 1 \mod 2}^{17} L_{ns}(c_{3,19},h_{1,i})\otimes(L(c_{8,19},h_{1,i})\oplus L(c_{8,19},h_{7,i})).
$$

Thus, the coset subalgebra of $L_{ns}(c_{3,19},0)$ is isomorphic to $L(c_{8,19},h_{1,1})\oplus L(c_{8,19},h_{7,1})$, a vertex superalgebra of type $(2,\frac{51}{2})$.

Finally, using decomposition formulas in Section 2.5, we get

$$L_{ns}\left(-\frac{11}{14},0\right)\otimes \mathcal{U}_{N=1}$$

decomposing as an $L_{ns}(c_{7,31},0)\otimes \mathcal{W}$-module, where $\mathcal{W}$ is vertex superalgebra with Virasoro decomposition:

$$\mathcal{W}=L(c_{12,31},0)\oplus L(c_{12,31},\frac{27}{2})\oplus L(c_{12,31},28)\oplus L(c_{12,31},\frac{145}{2}).$$

In particular, the even subalgebra $\mathcal{W}^{ev}$ is a W-algebra of type $(2,28)$, which seems to be new.

5. Computation of level 2 Urod vector

In this part we mainly follow [3] but we make everything slightly more explicit for the sake of computation.

5.1. **Automorphism.** We first recall some basic linear algebra facts discussed Section 2 of [3]. Let A be a semi-simple linear operator acting on the vertex algebra V such that $V=\bigoplus_{m\geq 0}V_m$, where V_m is the finite-dimensional eigenspace for A corresponding to eigenvalue $m\in Spec(A)$. Suppose that E is a locally finite operator on V, that is for every $v\in V$, the space $p(E)\cdot v$ spans a finite-dimensional subspace of V (here p runs through the set of polynomials). Define $\widehat{A}=A+E$. Next result is just a slightly more explicit version of what was discussed in loc.cit.

Proposition 5.1. *Let E be a graded linear map of degree k, that is $E:V_m\to V_{m+k}$ and E is of nilpotence index N_m* [4] *when restricted to V_m and $k\geq 1$. Let $\mathcal{B}_m$ be a basis V_m. Then $\widehat{A}$ is diagonalizable on V and*

$$\tilde{\mathcal{B}}_m=\{\tilde{v}:=v+\sum_{i=1}^{N_m-1}(-1)^i\frac{1}{k^i i!}E^i v.:v\in\mathcal{B}_m\}$$

is a basis of eigenvectors with eigenvalue m under $\widehat{A}$ and $V=\bigoplus_{m\geq 0}\tilde{V}_m$, where $\tilde{V}_m=Span(\tilde{B})$

In particular, if E is of degree 2, then

$$\tilde{\mathcal{B}}_m=\{\tilde{v}:=v+\sum_{i=1}^{N_m-1}(-1)^i\frac{1}{(2i)!!}E^i v.:v\in\mathcal{B}_m\}. \tag{21}$$

Going through each eigenspace V_m this way we obtain a linear isomorphism

$$v\to\tilde{v}\in\tilde{V}_m.$$

It is not hard to see [3] that this isomorphism is a vertex algebra automorphism.

Unfortunately, construction of the Urod conformal vector given in [3] is quite non-explicit so we have to dig deeper into the main construction. Following [3], we let $C=V_{\mathfrak{sl}_2}(k,0)\otimes L_{\mathfrak{sl}_2}(\ell,0)\otimes\Lambda^{\infty/2+*}(\mathfrak{n})$, where $\Lambda^{\infty/2+*}(\mathfrak{n})$ is generated by odd fields ψ and ψ^*. Letting $\widehat{A}(z)=h_2(z)+s^2J^{h_1}(z)e_2(z)$, yields

$$\widehat{A}_{(0)}=\underbrace{(h_2)_{(0)}}_{:=A}+\underbrace{(t^2J^{h_1}(z)e_2(z))_{(0)}}_{:=E}$$

[4] Observe that the local finiteness condition implies that E must be locally nilpotent.

where $J^{h_1}(z) = h_1(z) + 2{}^\circ_\circ\psi(z)\psi^*(z)^\circ_\circ$, and operator E is locally nilpotent with $k = 2$ (see Proposition 5.1), with respect to the $(h_2)_{(0)}$ grading. Using formula (21) we get a desired isomorphism $v \to \tilde{v} = \varphi_s(v)$. In particular, in our setting with $\ell = 2$, we have

(22)

$$\begin{aligned}
e_1(z) \mapsto &{}^\circ_\circ e_1(z)(1 - s^2 e_2(z) + \frac{1}{2}s^4 e_2(z)^2)^\circ_\circ \\
h_1(z) \mapsto &h_1(z) - ks^2\partial e_2(z) \\
f_1(z) \mapsto &{}^\circ_\circ f_1(z)(1 + s^2 e_2(z) + \frac{1}{2}s^4 e_2(z)^2)^\circ_\circ \\
e_2(z) \mapsto &e_2(z) \\
h_2(z) \mapsto &h_2(z) + s^2{}^\circ_\circ J^{h_1}(z)e_2(z)^\circ_\circ - \frac{k+2}{2}s^4{}^\circ_\circ(\partial e_2(z))e_2(z)^\circ_\circ \\
f_2(z) \mapsto &f_2(z) - \frac{1}{2}s^2({}^\circ_\circ J^{h_1}(z)h_2(z)^\circ_\circ + 2\partial J^{h_1}(z)) + \frac{1}{4}s^4(-{}^\circ_\circ J^{h_1}(z)^2 e_2(z)^\circ_\circ + (k+2){}^\circ_\circ h_2(z)\partial e_2(z)^\circ_\circ \\
&\quad + (k+2)\partial^2 e_2(z)) - \frac{1}{4}(k+2)s^6{}^\circ_\circ(\partial e_2(z))e_2(z)J^{h_1}(z)^\circ_\circ - \frac{1}{32}(k+2)^2 s^8{}^\circ_\circ(\partial^2 e_2(z))e_2(z)^2{}^\circ_\circ \\
\psi(z) \mapsto &{}^\circ_\circ\psi(z)(1 - s^2 e_2(z) + \frac{1}{2}s^4 e_2(z)^2)^\circ_\circ \\
\psi^*(z) \mapsto &{}^\circ_\circ\psi^*(z)(1 + s^2 e_2(z) + \frac{1}{2}s^4 e_2(z)^2)^\circ_\circ.
\end{aligned}$$

Equipped with this isomorphism we proceed as in loc.cit. The total Virasoro field for $H^\Delta_{DS}(L_k(\mathfrak{sl}_2) \otimes L_{sl_2}(2,0))$ is given by

$$\begin{aligned}
T_{\text{total}} = &\frac{1}{8}\left({}^\circ_\circ e_2(z)f_2(z)^\circ_\circ + {}^\circ_\circ f_2(z)e_2(z)^\circ_\circ + \frac{1}{2}{}^\circ_\circ h_2(z)^2{}^\circ_\circ\right) + \frac{1}{2k+4}\left({}^\circ_\circ e_1(z)f_1(z)^\circ_\circ + {}^\circ_\circ f_1(z)e_1(z)^\circ_\circ + \frac{1}{2}{}^\circ_\circ h_1(z)^2{}^\circ_\circ\right), \\
&+ {}^\circ_\circ(\partial\psi(z))\psi^*(z)^\circ_\circ + \frac{1}{2}\partial h_1(z) + \frac{1}{2}\partial h_2(z)
\end{aligned}$$

and the Virasoro field for $\mathcal{W}^k(\mathfrak{sl}_2)$ is

$$T_{DS}(z) = \frac{1}{k+2}\left(-J^{f_1}(z) + \frac{1}{4}{}^\circ_\circ J^{h_1}(z)^2{}^\circ_\circ + \frac{k+1}{2}\partial J^{h_1}(z)\right).$$

Finally, the $\ell = 2$ Urod conformal vector is given by $\varphi_s^{-1}(T_{\text{total}}(z)) - T_{DS}(z)$. Explicit computation shows that we have

$$\varphi_s^{-1}(T_{\text{total}}(z)) - T_{DS}(z) = T_{Urod}(z)\Big|_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to \frac{s^2(k+1)}{2}}}, \tag{23}$$

where T_{Urod} is as in Section 3. Thus we have proved that our family of conformal Urod structures, as a special case, contains the $\ell = 2$ Urod conformal vector constructed in [3].

6. Four fermion decomposition

In parallel with Section 3.3, where we investigated specific vertex superalgebras, we can now address the four-fermion model $\mathcal{F}^{\otimes 4} = \mathcal{F} \otimes \mathcal{U}_{N=1}$, using that $\mathcal{F} = L(c_{3,4}, 0) \oplus L(c_{3,4}, \frac{1}{2})$. The overall central charge is $c = -\frac{21}{2} + \frac{1}{2} = -10$. Using (20), we get two decompositions:

$$L(c_{3,4},0)\otimes\mathcal{F}^{\otimes^3}=\sum_{i=1 \mod 2}^{9}L_{ns}(c_{3,11},h_{1,i})\otimes L(c_{4,11},h_{1,i})$$

and

$$L(c_{3,4},\frac{1}{2})\otimes\mathcal{F}^{\otimes^3}=\sum_{i=1 \mod 2}^{9}L_{ns}(c_{3,11},h_{1,i})\otimes L(c_{4,11},h_{3,i}).$$

Combining these yields

$$\mathcal{F}^{\otimes^4}=\bigoplus_{i=1 \mod 2}^{9}L_{ns}(c_{3,11},h_{1,i})\otimes(L(c_{4,11},h_{1,i})\oplus L(c_{4,11},h_{3,i})).$$

Observe that the coset with respect to $SVir_{c_{3,11}}$ of $\mathcal{F}^{\otimes^4}$ is isomorphic to

$$L(c_{4,11},h_{1,1})\oplus L(c_{4,11},h_{3,1})=L(c_{4,11},0)\oplus L\left(c_{4,11},\frac{9}{2}\right),$$

which is a vertex operator superalgebra of type $(2,\frac{9}{2})$.

Four-fermion model can be also accessed from a different perspective because it belongs to the D-series due to $\mathfrak{so}(4)\cong\mathfrak{sl}_2\oplus\mathfrak{sl}_2$. Again the central charge is $c=-10$ and this exactly matches $c_{Urod,so(4)}$ given in Section 7.. More precisely, we have an isomorphism

$$\mathcal{F}^{\otimes^4}=L_{\mathfrak{sl}_2}(1,0)^{\otimes^2}\oplus L_{\mathfrak{sl}_2}(1,\omega_1)^{\otimes^2},$$

with respect to the standard grading, and thus

$$\mathcal{F}^{\otimes^4}=\mathcal{U}^{\otimes^2}\oplus L_{\mathfrak{sl}_2}(1,\omega_1)^{\otimes^2}.$$

where $\mathcal{U}$ is the $c=-5$ Urod algebra of $\mathfrak{sl}_2$ constructed in [4].

7. Five-fermion model and $\mathfrak{so}(5)$

Here we have $\mathcal{F}^{\otimes^5}=L_{\mathfrak{so}(5)}(1,0)\oplus L_{\mathfrak{so}(5)}(1,\omega_1)$, or we can use $\mathfrak{sp}(4)\cong\mathfrak{so}(5)$. Using [3, formula (33)] we found that the Urod central charge of $\mathfrak{so}(5)$ is given by:

$$c_{Urod,so(5)}=-\frac{10\ell(17+6\ell)}{3+\ell}$$

so for $\ell=1$ we get $c_{Urod,so(5)}=-\frac{115}{2}$. Narrowing down possible embeddings of rational W-(super)algebras using central charges and asymptotic dimensions suggests the following:

Conjecture 7.1. *Let $c(k)$ be a central charge from the minimal series of $\mathcal{W}_k(\mathfrak{so}(5))$. Also, let $\mathcal{F}^{\otimes^5}$ be equipped with an Urod conformal structure of level $\ell=1$. Then*

(a) The only conformal embedding of

$$\mathcal{W}_{k_1}(\mathfrak{so}_5)\otimes\mathcal{W}_{k_2}(\mathfrak{so}_5)\hookrightarrow L_{\mathfrak{so}(5)}(1,0)\hookrightarrow\mathcal{F}^{\otimes^5}$$

occurs for $c_1(k_1)=c_2(k_2)=-\frac{115}{4}$ and this embedding extends to a unique conformal embedding $\mathcal{W}_{k_1'}(\mathfrak{osp}(1|4))\otimes\mathcal{W}_{k_2'}(\mathfrak{osp}(1|4))\hookrightarrow\mathcal{F}^{\otimes^5}$, again with $c_1(k_1')=c_2(k_2')=-\frac{115}{4}$.

(b) The only embeddings of

$$\mathcal{W}_{k_1}(\mathfrak{osp}(1|4))\times\mathcal{W}_{k_2}(\mathfrak{so}_5)\hookrightarrow\mathcal{F}^{\otimes^5}$$

occur for: $c_1(k_1)=c_2(k_2)=-\frac{115}{4}$, $c_1(k_1)=-58, c_2(k_2)=\frac{1}{2}$ and $c_1(k_1)=-\frac{93}{2}, c_2(k_2)=-11$.

8. General $\mathcal{F}^{\otimes^n}$

Moving to higher ranks, we are no longer able to provide a complete (conjectural) statements as in Conjecture 7.1. Therefore, our focus shifts solely to decompositions directly associated with Urod algebras.

8.1. **Urod grading on $\mathcal{F}^{\otimes^n}$.** Let us first elucidate how the Urod grading on $\mathcal{F}^{\otimes^n}$ alters the standard grading, where all fermions are assigned a degree of $\frac{1}{2}$. This aspect was also briefly touched upon in the introduction, serving as motivation for Urod central charges of the orthogonal series.

Recall that the affine vertex algebra of $\mathfrak{so}(n)$, $n > 3$ at level 1 admits a fermionic construction [7]; see also [14] and [2] for related results. For $\mathfrak{so}(2n)$, $L_{\mathfrak{so}(2n)}(1,0)$ is realized as the even part of $\Lambda^{\otimes^{2n}}$, where $\Lambda^{\otimes^{2n}}$ is the infinite-wedge vertex algebra generated by n-pairs of *charged* free fermions $\psi_i(z), \psi_i^*(z)$, $1 \leq i \leq n$. Similarly, for $\mathfrak{so}(2n+1)$, the even part of the vertex superalgebra

$$\Lambda^{\otimes 2n} \otimes \mathcal{F} \cong \mathcal{F}^{\otimes^{2n+1}},$$

is isomorphic to $L_{\mathfrak{so}(2n+1)}(1,0)$, where $\mathcal{F}$ is as before a single neutral free fermion generated by $\varphi(z)$. Let us consider $\mathfrak{g} = \mathfrak{so}(2n+1)$ here; arguments for $\mathfrak{so}(2n)$ are very similar so we omit them.

The affine Lie algebra of $\mathfrak{so}(2n+1)$ is generated by the quadratic operators in the fields $\psi_i(z)$, $\psi_j^*(z)$ and $\varphi(z)$. The fields of the Chevalley generators of the Cartan algebra, $h_1(z)$,...,$h_n(z)$, are represented using fields ${}^\circ_\circ\psi_i(z)\psi_i^*(z)^\circ_\circ$, $1 \leq i \leq n-1$ and $2{}^\circ_\circ\psi_n(z)\psi_n^*(z)^\circ_\circ$. Let $\{e, f, h\}$ be an $\mathfrak{sl}_2$-triple where f is the principal nilpotent element with

$$h = \sum_{i=1}^{n-1}(2n-i+1)ih_i + \frac{n(n+1)}{2}h_n.$$

Therefore, to compute the conformal weights of $\psi_i(-1/2)\mathbf{1}$, $\psi_j^*(-1/2)\mathbf{1}$, and $\varphi(-1/2)\mathbf{1}$ with respect to the Urod conformal vector, we have to compute their eigenvalues with respect to $L(0)_{Sug} - (x_0)_{(0)}$ [3], where $x_0 = \frac{h}{2}$. Plugging the relevant quadratic fermionic generators into our formula for $h(z)$, together with $L(0)_{Sug}a(-1/2)\mathbf{1} = \frac{1}{2}a(-1/2)\mathbf{1}$, for $a = \varphi, \psi_i, \psi_i^*$, yields that the $(2n+1)$ fermions have the following conformal weights:

$$\left\{-\frac{2n-1}{2}, -\frac{2n-3}{2}, \cdots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \cdots \frac{2n+1}{2}\right\}.$$

With these explicit shifts, we can now compute the character of $\mathcal{F}^{\otimes^{2n+1}}$:

$$\mathrm{ch}[\mathcal{F}^{\otimes^{2n+1}}](q) = \prod_{i=0}^{2n}(-q^{\frac{-2n+1+2i}{2}}; q)_\infty = q^{-\sum_{i=1}^{n}\frac{i^2}{2}}(-q^{1/2}; q)_\infty^{2n+1},$$

as we previously argued in the introduction.

8.2. **$\mathcal{F}^{\otimes^{2n+1}}$.** Let us denote by $c_{\mathfrak{g}}$ the central charge of the principal affine W-algebra by $\mathcal{W}_k(\mathfrak{g})$. For orthogonal and symplectic series we have:

$$c_{sp(2n)} = -\frac{n(4n^2+4kn-2k-3)(2n(n+k+2)+k)}{k+n+1},$$

$$c_{osp(1|2n)} = -\frac{(2n+1)(2(2n-1)(k+n)-1)(4n(k+n)+1)}{2(2k+2n+1)},$$

$$c_{so(2n+1)} = -\frac{n(4n^2+2k(n+1)-3)(2n(2n+k-1)+k)}{k+2n-1},$$
$$c_{so(2n)} = -\frac{n\left(2kn-2k+4n^2-10n+5\right)\left(2kn-k+4n^2-8n+4\right)}{k+2n-2}.$$

The Feigin-Frenkel duality gives $\mathcal{W}_k(\mathfrak{so}(2n+1)) \cong \mathcal{W}_{-n-1+\frac{1}{2(k+2n-1)}}(\mathfrak{sp}_{2n}), n \geq 2$ so we can consider either algebra. A bit tedious computation using [3] shows that the central charge $c_{Urod,\mathfrak{g}}$ of the level ℓ Urod vertex algebra $\mathbb{L}_\ell = L_{\mathfrak{g}}(\ell, 0)$ is given by:

$$c_{Urod,so(2n+1)} = -\frac{\ell n(2n+1)(4n^2+2n-3+2\ell(1+n))}{2n+\ell-1},$$

$$c_{Urod,sp(2n)} = -\frac{\ell n(2n+1)(4n^2+2n-3+2\ell(n-1))}{n+\ell+1},$$

$$c_{Urod,so(2n)} = -\frac{\ell(2n^2-n)((2n-2)\ell+4n^2-8n+3)}{2n-2+\ell}.$$

In particular, these central charges are independent of the parameter k in [3, formula (33)]. Here, in the computations, we have chosen the same x_0 as in the definition of the principal W-algebra.

Using arguments from [3] and [5, Section 7] we see that the tensor product of affine W-superalgebras

$$\mathcal{W}_{-(n+1/2)+\frac{2n+k-1}{2(2k+n)}}(\mathfrak{osp}(1|2n)) \otimes \mathcal{W}_{-(2n-1)+\frac{2n-1}{(2(1+n)(k+2n))}}(\mathfrak{so}(2n+1))$$

conformally embeds into the vertex superalgebra

$$\mathcal{W}_k(\mathfrak{so}(2n+1)) \otimes \mathcal{F}^{\otimes^{2n+1}},$$

where $\mathcal{F}^{\otimes^{2n+1}}$ is equipped with an Urod conformal structure. If we choose $k = -(2n-1) + \frac{2n+1}{2n+2}$, which belong to the minimal series, we have $\mathcal{W}_k(\mathfrak{so}(2n+1)) \cong \mathbb{C}$, and hence

Proposition 8.1. *There is a conformal embedding:*

$$\mathcal{W}_{-(n+\frac{1}{2})+\frac{(n+\frac{1}{2})}{3+4n}}(\mathfrak{osp}(1|2n)) \otimes \mathcal{W}_{-(2n-1)+\frac{2n-1}{3+4n}}(\mathfrak{so}(2n+1)) \hookrightarrow \mathcal{F}^{\otimes^{2n+1}}.$$

Next discussion does not seem to follow the setup of [3]. First we observe a result from [5, Section 7]:

Lemma 8.2. *The rational vertex algebra $\mathcal{W}_{-(n+1)+\frac{1+2n}{8n}}(\mathfrak{sp}_{2n}) \cong \mathcal{W}_{-(2n-1)+\frac{4n}{1+2n}}(\mathfrak{so}(2n+1))$, of central charge $\frac{1}{2}c_{Urod,so(2n+1)}$, is the even part of the affine vertex superalgebra $\mathcal{W}_\ell(\mathfrak{osp}(1|2n))$ with $\ell = -(n+\frac{1}{2}) + \frac{2n-1}{8n}$. Consequently, $\mathcal{W}_\ell(\mathfrak{osp}(1|2n))$ is rational.*

Then our next conjecture can be viewed as a higher-rank generalization of Theorem 3.3:

Conjecture 8.3. *(Square root [5] of $\mathcal{F}^{\otimes^{2n+1}}$) There is a conformal embedding*

$$\mathcal{W}_{-(n+\frac{1}{2})+\frac{2n-1}{8n}}(\mathfrak{osp}(1|2n))^{\otimes 2} \hookrightarrow \mathcal{F}^{\otimes^{2n+1}}.$$

[5] If V and W are rational, and $W \otimes W$ conformally embedded in V, we say that W is a square root V.

8.3. $\mathcal{F}^{\otimes^{2n}}$. The case $\mathcal{F}^{\otimes^{2n}}$ is easier because this is a level 1 module for the simply-laced affine Lie algebra of type $D_n^{(1)}$, $n \geq 2$, so results from [3] are applicable. We use [7]

$$\mathcal{F}^{\otimes 2n} = L_{\mathfrak{so}(2n)}(1,0) \oplus L_{\mathfrak{so}(2n)}(1,\omega_1),$$

so that $\mathcal{F}^{\otimes 2n}$ can be equipped with an Urod structure. Using [3, Corollary 8.8] we get an embedding

$$\mathcal{W}_{-(2n-2)+\frac{4n-3}{2n-2}}(\mathfrak{so}(2n)) \otimes \mathcal{W}_{-(2n-2)+\frac{4n-3}{2n-1}}(\mathfrak{so}(2n)) \hookrightarrow L_{\mathfrak{so}(2n)}(1,0)$$

and similarly for $L_{\mathfrak{so}(2n)}(1,\omega_1)$. However, the W-algebra modules that appear in the decomposition of $L_{\mathfrak{so}(2n)}(1,\omega_1)$ are different from those in $L_{\mathfrak{so}(2n)}(1,0)$ (see again [3, Corollary 8.8.2]) so this way we do not obtain new super extensions of $\mathcal{W}_k(\mathfrak{so}(2n))$.

9. Explicit formulas

In this section we give explicit formulas for commuting Virasoro and super-Virasoro generators within $V(c_t,0)\otimes\mathcal{U}_{N=1}$. In fact this is slightly more general than the setup of [3] because we provide a two-parameter family of decompositions. Inside of $V(c_t,0)\otimes\mathcal{U}_{N=1}$ we have commuting copies of Vir_{2+t} and $SVir_{1+\frac{2}{t}}$ generated by

$$\begin{aligned}
T_{c^{\mathrm{Vir}}_{2+t}}(z) = &-\frac{1}{\epsilon}f(z) + \eta {}^\circ_\circ e(z)f(z){}^\circ_\circ - \frac{4\eta+2\eta t-1}{4(t+2)}{}^\circ_\circ h(z)^2{}^\circ_\circ - \frac{4\eta+2\eta t-t-1}{2(t+2)}\partial h(z)\\
&+\frac{\eta(6\eta+3\eta t-2)}{4(t+2)}\epsilon{}^\circ_\circ h(z)^2e(z){}^\circ_\circ + \frac{\eta(2\eta+\eta t-t)}{2(t+2)}\epsilon{}^\circ_\circ(\partial h(z))e(z){}^\circ_\circ\\
&-\frac{16\eta^2-8\eta+4\eta^2t^2+2\eta t^2+16\eta^2 t-t}{4(t+2)^2}\epsilon{}^\circ_\circ h(z)\partial e(z){}^\circ_\circ + \frac{t(8\eta+4\eta t+2t-1)}{4(t+2)^2}\epsilon\partial^2 e(z)\\
&+\frac{-2\eta^3(t+2)^3+2\eta^2(t+2)^2(2t+1)+2\eta t(t+2)(2t-1)+(2-3t)t}{4(t+2)^3}\epsilon^2{}^\circ_\circ(\partial^2 e(z))e(z){}^\circ_\circ\\
&+\frac{-2t^2-2\eta^3(t+2)^3+2\eta^2(t+2)^2(2t+1)+\eta t(t+2)(4t-1)+t}{4(t+2)^3}\epsilon^2{}^\circ_\circ(\partial e(z))^2{}^\circ_\circ\\
&+\frac{6\eta^3(t+2)^3-6\eta^2(t+2)^2-\eta t(t+2)+t}{8(t+2)^3}\epsilon^2{}^\circ_\circ(\partial h(z))e(z)^2{}^\circ_\circ\\
&+\frac{t\left(-16\eta\left(t^2+t-2\right)-8\eta^2(t+2)^2+11t-12\right)}{32(t+2)^4}\epsilon^3{}^\circ_\circ(\partial^2 e(z))e(z)^2{}^\circ_\circ\\
&+\frac{t}{t+2}T_{c^{\mathrm{Vir}}_t}(z) - \frac{t}{(t+2)^2}\epsilon{}^\circ_\circ e(z)T_{c^{\mathrm{Vir}}_t}(z){}^\circ_\circ + \frac{t-\eta t(t+2)}{(t+2)^3}\epsilon^2{}^\circ_\circ e(z)^2T_{c^{\mathrm{Vir}}_t}(z){}^\circ_\circ
\end{aligned}$$

$$
\begin{aligned}
T_{c^{N=1}_{1+\frac{2}{t}}}(z) =& \frac{1}{\epsilon} f(z) + \frac{1}{4}(1-4\eta){}^\circ_\circ e(z)f(z){}^\circ_\circ + \frac{8\eta(t+2)+t-2}{16(t+2)}{}^\circ_\circ h(z)^2{}^\circ_\circ + \frac{16\eta+8\eta t - t+2}{8(t+2)}\partial h(z)\\
&+ \frac{\eta(2-3\eta(t+2))}{4(t+2)}\epsilon{}^\circ_\circ h(z)^2 e(z){}^\circ_\circ + \frac{\eta(t-\eta(t+2))}{2(t+2)}\epsilon{}^\circ_\circ(\partial h(z))e(z){}^\circ_\circ\\
&+ \frac{16\eta^2-8\eta+4\eta^2t^2+2\eta t^2+16\eta^2 t - t}{4(t+2)^2}\epsilon{}^\circ_\circ h(z)\partial e(z){}^\circ_\circ + \frac{8\eta + t(3-2\eta t)-4}{4(t+2)^2}\epsilon\partial^2 e(z)\\
&+ \frac{2\eta^3(t+2)^3 - 2\eta^2(t-1)(t+2)^2 + 2\eta(t+2)(3t-4)+(t-2)^2}{4(t+2)^3}\epsilon^2{}^\circ_\circ(\partial^2 e(z))e(z){}^\circ_\circ\\
&+ \frac{(\eta(t+2)-1)\left(-2\eta\left(t^2-4\right)+2\eta^2(t+2)^2+3t-4\right)}{4(t+2)^3}\epsilon^2{}^\circ_\circ(\partial e(z))^2{}^\circ_\circ\\
&+ \frac{-6\eta^3(t+2)^3+6\eta^2(t+2)^2+\eta t(t+2)-t}{8(t+2)^3}\epsilon^2{}^\circ_\circ(\partial h(z))e(z)^2{}^\circ_\circ\\
&+ \frac{t\left(16\eta\left(t^2+t-2\right)+8\eta^2(t+2)^2-11t+12\right)}{32(t+2)^4}\epsilon^3{}^\circ_\circ(\partial^2 e(z))e(z)^2{}^\circ_\circ\\
&+ \frac{2}{t+2}T_{c_t^{\mathrm{Vir}}}(z) + \frac{t}{(t+2)^2}\epsilon{}^\circ_\circ e(z)T_{c_t^{\mathrm{Vir}}}(z){}^\circ_\circ + \frac{t(\eta(t+2)-1)}{(t+2)^3}\epsilon^2{}^\circ_\circ e(z)^2 T_{c_t^{\mathrm{Vir}}}(z){}^\circ_\circ
\end{aligned}
$$

$$
\begin{aligned}
G_{c^{N=1}_{1+\frac{2}{t}}}(z) =& \frac{\eta\epsilon(\eta(t+2)+t-2)}{2\sqrt{t(t+2)}}{}^\circ_\circ(\partial h(z))\varphi_{-\frac{1}{2}}(z){}^\circ_\circ\\
&- \frac{\sqrt{t(t+2)}\epsilon^2\left(2\eta^2(t+2)^2-\eta(t+2)+t-1\right)}{4(t+2)^3}{}^\circ_\circ e(z)^2 f(z)\varphi_{-\frac{1}{2}}(z){}^\circ_\circ\\
&- \frac{2\eta(t+2)+t-2}{2\sqrt{t(t+2)}}{}^\circ_\circ h(z)\varphi_{\frac{1}{2}}(z){}^\circ_\circ + \frac{\eta\epsilon(\eta(t+2)-2)}{2\sqrt{t(t+2)}}{}^\circ_\circ h(z)e(z)\varphi_{\frac{1}{2}}(z){}^\circ_\circ\\
&+ \frac{\sqrt{t(t+2)}\epsilon^2(\eta(t+2)(4\eta(t+2)-3)+1)}{4(t+2)^3}{}^\circ_\circ h(z)e(z)^2\varphi_{\frac{1}{2}}(z){}^\circ_\circ\\
&+ \frac{t^2\epsilon(1-2\eta(t+2))}{2(t(t+2))^{3/2}}{}^\circ_\circ(\partial e(z))\varphi_{\frac{1}{2}}(z){}^\circ_\circ\\
&- \frac{\sqrt{t(t+2)}\epsilon^3\left(16\eta^3(t+2)^3-16\eta^2(t+2)^2+8\eta t(t+2)-5t+4\right)}{16(t+2)^4}{}^\circ_\circ(\partial e(z))e(z)^2\varphi_{\frac{1}{2}}(z){}^\circ_\circ\\
&+ \frac{1-\eta(t+2)}{\sqrt{t(t+2)}}{}^\circ_\circ e(z)\varphi_{\frac{3}{2}}(z){}^\circ_\circ + \frac{\sqrt{t(t+2)}}{t\epsilon}\varphi_{\frac{3}{2}}(z) + \frac{t^2\epsilon}{(t(t+2))^{3/2}}{}^\circ_\circ T(z)\varphi_{-\frac{1}{2}}(z){}^\circ_\circ
\end{aligned}
$$

Here we have

$$
T_{c_{2+t}^{\mathrm{Vir}}}(z) + T_{c^{N=1}_{1+\frac{2}{t}}}(z) = T_{c_t}(z) + T_{Urod}(z)\Big|_{\substack{\epsilon_1\to\frac{\epsilon^2\left(2\eta\left(t^2+t-2\right)+\eta^2(t+2)^2-t+1\right)}{2(t+2)^2}\\ \epsilon_2\to\frac{\epsilon(\eta(t+2)+t-1)}{2(t+2)}}}.
$$

10. Future work

Based on the ideas and results in our paper (and of course [3]), there are several future avenues for exploration.

(Twisted modules) Clearly, we can modify many results here and in [3] for twisted modules with respect to the parity operators σ. In particular, all results for $\mathcal{U}_{N=1}$ in Section 3 have a version where the Neveu-Schwarz algebra is replaced with the Ramond superconformal algebra. For example, the σ-twisted $\mathcal{F}^{\otimes^2}$-module $\mathcal{F}_{\mathcal{R}}^{\otimes^2}$ admits an isomorphism

$$\mathcal{F}_{\mathcal{R}}^{\otimes^2} \cong L_{\mathcal{R}}\left(-11, -\frac{3}{8}\right).$$

(Nilpotent elements) Another direction, motivated by [3], is to study different gradings and decompositions on $\mathcal{F}^{\otimes^n}$ coming from non-principal nilpotent elements. As an example, here we consider $f = f_{subreg}$ (subregular) and $f = f_{min}$ (minimal) nilpotent elements of $\mathfrak{so}(5)$.

The Urod structure on the level $\ell \in \mathbb{N}$ affine vertex algebra $\mathbb{L}_\ell = L_{\mathfrak{so}(5)}(\ell, 0)$ has central charge $c = -\frac{2\ell(13+6\ell)}{3+\ell}$. Letting $\ell = 1$, gives conformal vector $T_{Urod, f_{subreg}}$ of central charge $-\frac{19}{2}$ and with respect to the degree operator fermionic generators receive conformal weights: $-\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$. Consequently,

$$\mathrm{ch}_{f_{subreg}}[\mathcal{F}^{\otimes^5}](q) = q^{-1/2}(-q^{1/2}; q)_\infty^5.$$

For $f = f_{min}$, the Urod affine vertex algebra $\mathbb{L}_\ell = L_{\mathfrak{so}(5)}(\ell, 0)$ has central charge $c = -\frac{2\ell(4+3\ell)}{3+\ell}$. Similarly, letting $\ell = 1$, gives central charge $-\frac{7}{2}$ and fermionic generators have conformal weights: $0, 0, 1, 1, \frac{1}{2}$, and therefore

$$\mathrm{ch}_{f_{min}}[\mathcal{F}^{\otimes^5}](q) = 4(-q; q)_\infty^4(-q^{1/2}; q)_\infty.$$

Observe that in both cases $q^{-c/24}\mathrm{ch}_f[\cdot]$ are modular functions.

Acknowledgment

The first author was partially supported on the NSF grant 2101844. We would like to thank the referee for useful comments.

References

[1] Adamović, Dražen, "Rationality of Neveu-Schwarz vertex operator super- algebras", Internat. Math. Res. Notices 1997 (1997), 865–874.

[2] Adamović, Dražen, and Ozren Perše. "On coset vertex algebras with central charge 1." Mathematical Communications 15.1 (2010): 143-157.

[3] Arakawa, Tomoyuki, Thomas Creutzig, and Boris Feigin. "Urod algebras and translation of W-algebras." Forum of Mathematics, Sigma. Vol. 10. Cambridge University Press, 2022.

[4] Bershtein, Mikhail, Boris Feigin, and Alexei Litvinov. "Coupling of two conformal field theories and Nakajima-Yoshioka blow-up equations." arXiv preprint arXiv:1310.7281 (2013).

[5] Creutzig, Thomas, and Andrew R. Linshaw. "Trialities of orthosymplectic W-algebras." Advances in Mathematics 409 (2022): 108678.

[6] Feigin, Boris, and Sergei Gukov. "VOA [M4]." Journal of Mathematical Physics 61.1 (2020).

[7] Frenkel, Igor B. "Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory." Journal of functional analysis 44.3 (1981): 259-327.

[8] Iohara, Kenji, and Yoshiyuki Koga. Representation theory of the Virasoro algebra. Springer Science & Business Media, 2010.

[9] Iohara, Kenji, and Yoshiyuki Koga. "Representation theory of Neveu–Schwarz and Ramond algebras I: Verma modules." Advances in Mathematics 178.1 (2003): 1-65.

[10] Kac, Victor G. Vertex Algebras for Beginners. No. 10. American Mathematical Soc., 1998.
[11] Goddard, Peter, Adrian Kent, and David Olive. "Unitary representations of the Virasoro and super-Virasoro algebras." Communications in Mathematical Physics 103 (1986): 105-119.
[12] Kac, Victor G., and Minoru Wakimoto. "Classification of modular invariant representations of affine algebras." Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988) 7 (1989): 138-177.
[13] Kac, Victor G., and Minoru Wakimoto. "Modular invariant representations of infinite-dimensional Lie algebras and superalgebras." Proceedings of the National Academy of Sciences 85.14 (1988): 4956-4960.
[14] Lepowsky, James, and Mirko Primc. "Standard modules for type one affine Lie algebras." Number Theory: A Seminar held at the Graduate School and University Center of the City University of New York 1982. Berlin, Heidelberg.

Department of Mathematics and Statistics, SUNY-Albany
Email address: `amilas@albany.edu`

Mathematics Department, Randolph College
Email address: `mpenn@randolphcollege.edu`